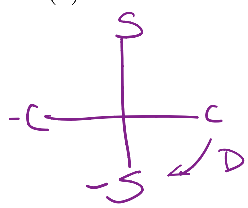
Calc BC 9.10 Notes Name: Block: Seat:

Find the radius (and interval) of convergence for

1. Use the function $f(x) = \cos x$ to form the Maclaurin series
$$\sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} x^n$$

   (a) Find the derivatives



$f(x) = \cos x$
$f'(x) = -\sin x$
$f''(x) = -\cos x$
$f'''(x) = \sin x$ etc

   (b) evaluate the derivatives at zero

$f(0) = 1$
$f'(0) = 0$
$f''(0) = -1$
$f'''(0) = 0$ etc

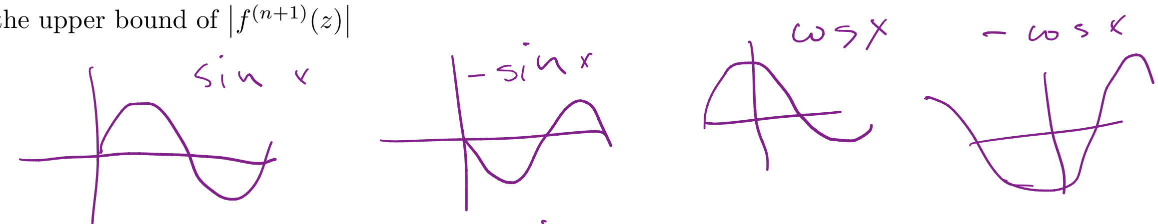
   (c) Assemble the series

$$\cos x = 1 - \frac{x^2}{2} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots + \frac{(-1)^n x^{2n}}{(2n)!} - \cdots$$

   (d) Use the ratio test to conclude that the series converges for all x

$$\lim_{n\to\infty} \left| \frac{x^{2n+2}}{(2n+2)!} \cdot \frac{(2n)!}{x^{2n}} \right| = \lim_{n\to\infty} \left| \frac{x^2}{(2n+2)(2n+1)} \right| < 1$$
for all $x \in \mathbb{R}$

   (e) What is the upper bound of $\left|f^{(n+1)}(z)\right|$



Max is $y = 1$

2. Show that the Maclaurin series for $f(x) = \cos x$ converges to $\cos x$ for all x (Use Th 9.19, and show the remainder is zero by squeezing it between 0 and $\frac{|x|^{n+1}}{(n+1)!}$)

$$\cos x = 1 - \frac{x^2}{2} + \frac{x^4}{4!} - \cdots \frac{(-1)^n x^{2n}}{(2n)!} + \cdots$$

Because $f^{(n+1)}(x) = \pm \cos x$

and $f^{(n+1)}(x) = \pm \sin x$

$$\left| f^{(n+1)}(z) \right| \leq 1$$

for all $z \in \mathbb{Z}$

Using Th 9.19 (Taylor Remainder Thm)

$$0 \leq \left| R_n(x) \right| = \left| \frac{f^{(n+1)}(z)\, x^{n+1}}{(n+1)!} \right| \leq \frac{|x|^{n+1}}{(n+1)!}$$

Since $\lim_{n \to \infty} \frac{|x|^{n+1}}{(n+1)!} = 0$

$$0 \leq \left| R_n(x) \right| \leq 0$$

$$R_n(x) = 0$$

as $n \to \infty$

by Squeeze Thm

3. Find the Maclaurin series for $f(x) = \cos x^2$

   Instead of finding the derivatives, and finding $f'(0), f''(0), \ldots$, consider the known power series for $g(x) \cos x$. since $f(x) = g(x^2)$ we have only to substitute to obtain the power series.

$$\cos x = 1 - \frac{x^2}{2} + \frac{x^4}{4!} - \cdots \frac{(-1)^n x^{2n}}{(2n)!} + \cdots$$

$$\cos x^2 = 1 + \frac{x^4}{2} + \frac{x^8}{4!} - \frac{x^{12}}{6!} + \cdots - \frac{(-1)^n (x^2)^{2n}}{(2n)!} + \cdots$$

$$\cos x^2 = \sum_{n=0}^{\infty} \frac{(-1)^n (x)^{4n}}{(2n)!}$$

4. Find the Maclaurin series for $f(x) = (1+x)^k$ (aka Binomial series) p673

(a) Find the derivatives

$$f(x) = (1+x)^k$$
$$f'(x) = k(1+x)^{k-1}$$
$$f''(x) = k(k-1)(1+x)^{k-2}$$
$$\vdots$$
$$f^{(n)}(x) = k(k-1)\ldots(k-n+1)(1-x)^{k-n}$$

(b) evaluate the derivatives at zero

$$f(0) = 1$$
$$f'(0) = k$$
$$f''(0) = k(k-1)$$
$$f'''(0) = k(k-1)(k-2)$$
$$f^{(n)}(0) = k(k-1)\cdots(k-n+1)$$

(c) Assemble the series

$$1 + kx + \frac{k(k-1)x^2}{2} + \frac{k(k-1)(k-3)x^3}{3!} + \cdots + \frac{k(k-1)\cdots(k-n+1)x^n}{n!} + \ldots$$

(d) Use the ratio test to the radius and interval of convergence

$$\lim_{n\to\infty}\left|\frac{k(k-1)\cdots(k-n+1)(k-n+2)}{(n+1)n!}\cdot\frac{n!}{k(k-1)\cdots(k-n+1)}\right|$$

$$\lim_{n\to\infty}\left|\frac{k-n+2}{n+1}\right| < 1$$

$$-1 < k < 1$$

5. Using the known series for the basic functions find the Maclaurin series for $f(x) = \sqrt[3]{1+x}$

$$(1+x)^k = 1 + kx + \frac{k(k-1)x^2}{2!} + \cdots$$

Let $k = \frac{1}{3}$

$$(1+x)^{1/3} = 1 + \frac{x}{3} - \frac{2x^2}{3^2 2!} + \frac{2 \cdot 5 x^3}{3^3 \cdot 3!} - \frac{2 \cdot 5 \cdot 8 x^4}{3^4 4!} + \cdots$$

$$P_4(x) = 1 + \frac{x}{3} - \frac{x^2}{9} + \frac{5x^3}{81} - \frac{10x^4}{243}$$

6. Using the known series for the basic functions find the Maclaurin series for $f(x) = \cos(\sqrt{x})$

$$\cos x = 1 + \frac{x^2}{2!} + \frac{x^4}{4!} + \cdots$$

$$\cos x^{1/2} = 1 + \frac{x^1}{2!} + \frac{x^2}{4!}$$

7. Using the known series for the basic functions find the first three non-zero terms of the Maclaurin series for $f(x) = e^x \arctan x$

$$e^x \arctan x = \left(1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots\right)\left(x - \frac{x^3}{3} + \frac{x^5}{5} + \cdots\right)$$

$$= x + x^2 + \frac{1}{6}x^3 - \frac{1}{6}x^4 + \frac{3}{40}x^5 + \cdots$$

8. Using the known series for the basic functions find the first three non-zero terms of the Maclaurin series for $f(x) = \sin^2 x = \dfrac{1 - \cos(2x)}{2}$ $= \frac{1}{2} - \frac{1}{2}\cos 2x$

$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} + \frac{x^6}{6!} + \frac{x^8}{8!}$

$\cos 2x = 1 - \frac{2^2}{2!}x^2 - \frac{2^4 x^4}{4!} + \frac{2^6 x^6}{6!} + \cdots$

$-\frac{1}{2}\cos 2x = -\frac{1}{2} + \frac{2x^2}{2!} - \frac{2^3}{4!}x^4 + \frac{2^6 x^6}{6!} + \cdots$

$\frac{1}{2} - \frac{1}{2}\cos 2x = \frac{1}{2} - \frac{1}{2} + \frac{2x^2}{2!} - \frac{2^3 x^4}{4!} + \frac{2^5 x^6}{6!} \cdots$

so $\sin^2 x = \frac{2x^2}{2!} - \frac{2^3 x^4}{4!} + \frac{2^5 x^6}{6!} - \frac{2^7 x^8}{8!}$

$R = \infty$

9. Use a power series to approximate

$$\int_0^1 e^{-x^2}\,dx$$

with an error of less than 0.01.

(a) Use the power series for e^x, replacing x with $-x^2$

(b) Using the Alternating series Remainder Th, which term is less than 0.01?

(c) So you should add how many terms?

(d) Your estimate is then...

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \cdots + \frac{x^n}{n!} + \cdots$$

$$e^{-x^2} = 1 - x^2 + \frac{x^4}{2!} - \frac{x^6}{3!} + \frac{x^8}{4!} - \cdots$$

$$\int_0^1 e^{-x^2}\,dx = \left[x - \frac{x^3}{3} + \frac{x^5}{5\cdot 2!} - \frac{x^7}{7\cdot 3!} + \cdots\right]_0^1$$

$$= 1 - \frac{1}{3} + \frac{1}{10} - \frac{1}{42} + \frac{1}{216} - \cdots$$

$$P_4 = 1 - \frac{1}{3} + \frac{1}{10} - \frac{1}{42} \approx .74$$

so by AST Rem Th.

$$\left| \int_0^1 e^{-x^2}\,dx - P_4 \right| < \frac{1}{216} \quad (\approx .005)$$

Answers:

4. R = 1, (-1,1)

9. the fifth term is $\frac{1}{216} < \frac{1}{100}$, so use the sum of the first 4 terms to estimate 0.74